Graphical Definition of a Limit

Given a graph, we define the following idea of a limit:

A limit is the idea of expressing the y value as we approach a particular x value in the following ways:

$\lim_{x \to a^+} f(x) = L$ This says "As we approach the x value of a from the right side of the function $f(x)$, the function goes towards the y value of L.

$\lim_{x \to a^-} f(x) = L$ This says "As we approach the x value of a from the left side of the function $f(x)$, the function goes towards the y value of L.

If the left side and right side match up, we use the following notation:

$\lim_{x \to a} f(x) = L$ This says "As we approach the x value of a from the both sides of the function $f(x)$, the function goes towards the y value of L.

Definition of a Limit

Definition: The function $f(x)$ has the limit L as x approaches a, written

$$\lim_{x \to a} f(x) = L$$

if the value of $f(x)$ can be made as close to the number L as we please by taking x sufficiently close to (but not equal to) a.

Note that the function may tend to a number, ∞ or $-\infty$.

It is also possible that a function does not tend to anything, and in this case we say the limit does not exist.

We also note that $\lim_{x \to a} f(x) = L$ means we approach (and that we do not need to reach the a value for x) where as $f(a)$ is asking for the point when we reach $x = a$ (and does not care what happens as we approach the value).

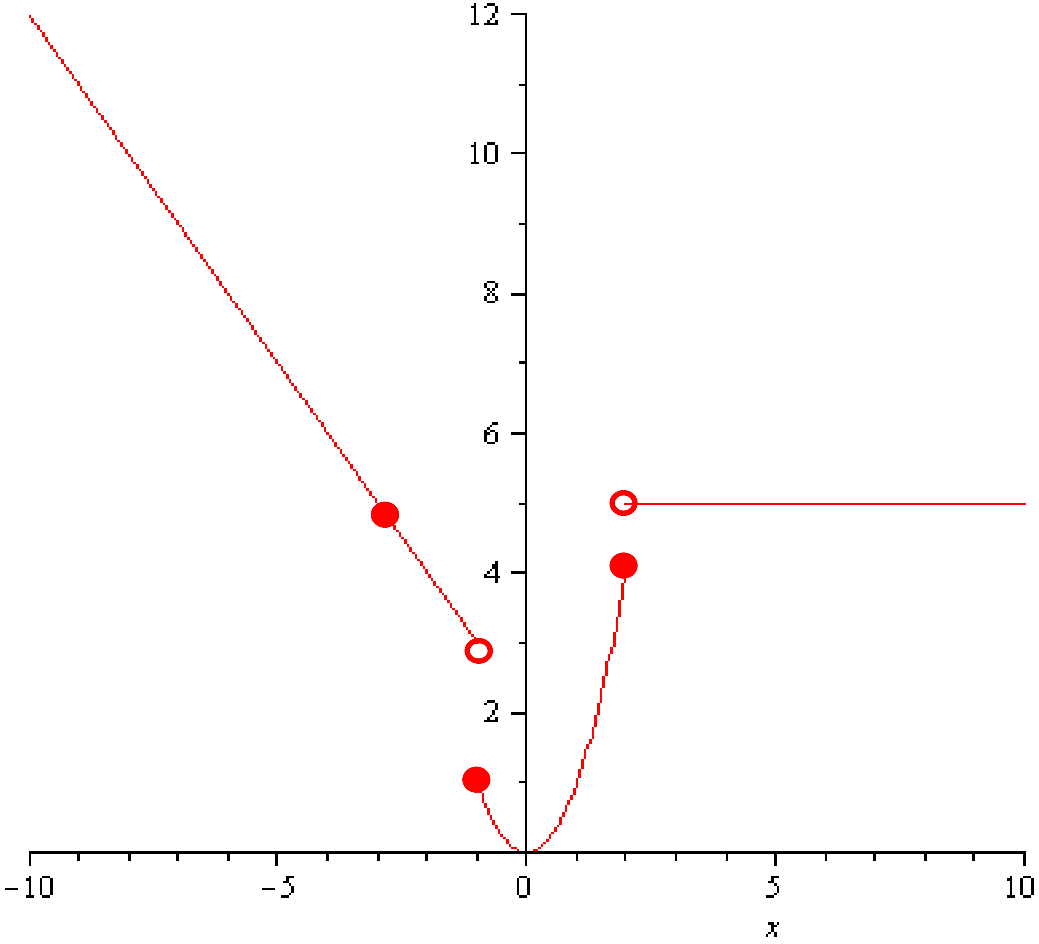
Strategy: Finding Limits Using Graphs

How To Use it:	When To Use it:	Why this works?
To find a limit from one side a^+ or a^- 1) Pick a point close a on the right side of a (when doing a^+) or on the left side of a (when doing a^-). 2) Trace the graph until you arrive at $x = a$ (note that you are not allowed to jump when arriving at a. 3) This will be the value of the limit. 4) Note: to find the limit when $x \to a$, you simply doing both a^+ and a^- and see if the limits match. If they do, this is the value of the limit. If they do not, the limit does not exist. To find a limit when $x \to \infty$ we choose a point on the far right of the graph and trace the graph until you see where the function is going (either it gets stuck at a value L, it shoots down to $-\infty$, shoots up to ∞, or bounces back and forth and does not exist). For $x \to -\infty$, it is the same idea except trace the graph to the left instead.	Finding limits given a graph.	This is simply using a systematic way of finding limits off of the definition.

Examples: Using Limits on Graphs

Example 1:

From the graph below, determine the requested values:



$\lim_{x\to 2^+} f(x) =$ 5

$\lim_{x\to 2^-} f(x) =$ 4

$\lim_{x\to 2} f(x) =$ DNE

$f(2) =$ 4

$\lim_{x\to -3^+} f(x) =$ 5

$\lim_{x\to -3^-} f(x) =$ 5

$\lim_{x\to -3} f(x) =$ 5

$f(-3) =$ 5

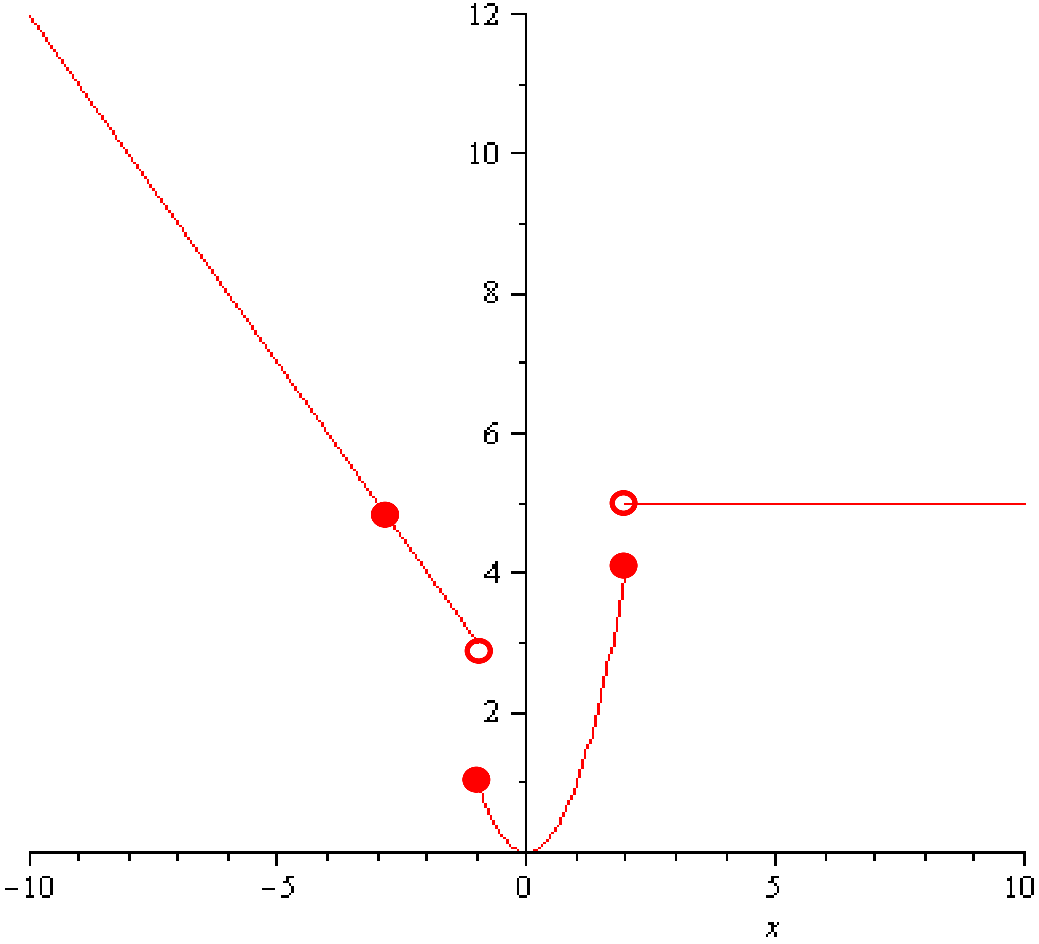
$\lim_{x\to \infty} f(x) =$ 5

$\lim_{x\to -\infty} f(x) =$ ∞

Examples: Using Limits on Graphs

Example 2:

From the graph below, determine the requested values:



4 limits with a value of 4

$\lim_{x \to -2^-} f(x) = 4$ $\lim_{x \to -2^+} f(x) = 4$

$\lim_{x \to -2} f(x) = 4$ $\lim_{x \to 2^-} f(x) = 4$

2 limits that would be undefined

$\lim_{x \to -1} f(x) =$DNE $\lim_{x \to 2} f(x) =$ DNE

A limit that goes to 0

$\lim_{x \to 0^-} f(x) = 0$ $\lim_{x \to 0^+} f(x) = 0$ $\lim_{x \to 0} f(x) = 0$

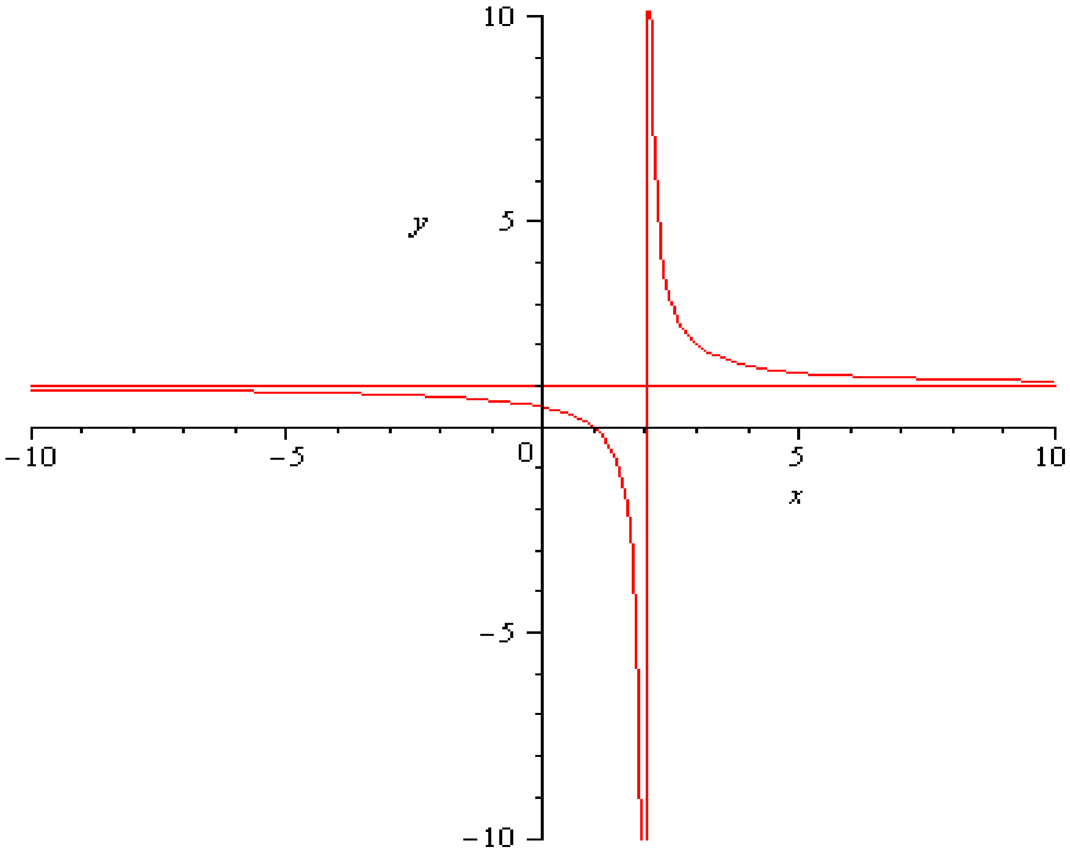
A limit that goes to ∞

$\lim_{x \to -\infty} f(x) = \infty$

Examples: Using Limits on Graphs

Example 3:

From the graph below, determine the requested values:



$\lim_{x \to 2^+} f(x) =$ ∞

$\lim_{x \to 2^-} f(x) =$ $-\infty$

$\lim_{x \to 2} f(x) =$ DNE

$f(2) =$ DNE

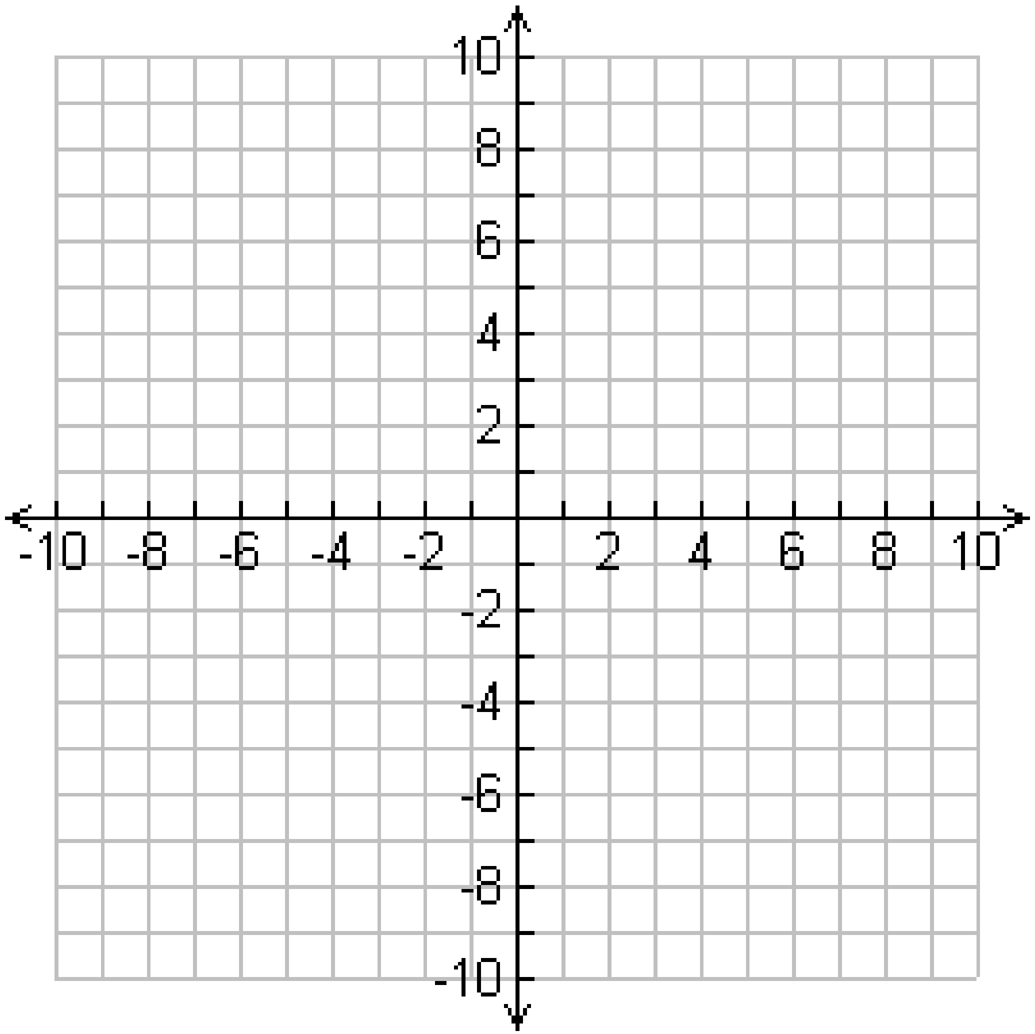
$\lim_{x \to \infty} f(x) =$ 1

$\lim_{x \to -\infty} f(x) =$ 1

Examples: Using Limits on Graphs

Example 4:

Given the information below, create a sketch of a graph of the function:



$\lim_{x\to 1^+} f(x) = 4$ $\lim_{x\to 1^-} f(x) = 2$ $f(1) = -2$

$\lim_{x\to 3} f(x) = 5$ $f(3) = 0$

$\lim_{x\to -2^-} f(x) = \infty$ $\lim_{x\to -2^+} f(x) = -\infty$

$\lim_{x\to \infty} f(x) = \infty$ $\lim_{x\to -\infty} f(x) = 2$

Properties of Limits

If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then

1. $\lim_{x\to a}(f(x))^r = \left(\lim_{x\to a} f(x)\right)^r = L^r, r \in\Re$

2. $\lim_{x\to a} cf(x) = c\lim_{x\to a}f(x) = cL, c \in\Re$

3. $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = L \pm M$

4. $\lim_{x\to a}[f(x).\, g(x)] = \left(\lim_{x\to a} f(x).\right)\left(\lim_{x\to a} g(x)\right) = LM$

5. $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{M}$ provided $M \neq 0$

Strategy: Finding Limits using Direct Substitution

How To Use it:	When To Use it:	Why this works?
When asked to find $\lim_{x \to a} f(x)$, attempt to substitute the a value into the function. If the value can be calculated (ie, it turns out to be a number without issues like $\sqrt{-}$, $\div 0$, $\log(0)$, etc…), then this number is the value of the limit.	This is the first strategy that should be attempted as it is easy to calculate.	When a function is continuous (that is you can draw the graph with one continuous motion through the point) the value of the limit is equal to the value of the function. That is $\lim_{x \to a} f(x) = f(a)$.

Example 1:

Determine the limits below:

a) $\lim_{x \to 2} 3x + 1$

b) $\lim_{x \to 1} x^2 - \frac{1}{x} + 1$

Solution:

a) $\lim_{x \to 2} 3x + 1 \quad = 3(2) + 1$

$= 7$

b) $\lim_{x \to 1} x^2 - \frac{1}{x} + 1 \quad = 1^2 - \frac{1}{1} + 1$

$= 1$

Strategy: Finding Limits using Factoring

How To Use it:	When To Use it:	Why this works?
When you try direct substitution and the result is $\frac{0}{0}$ we call this an indeterminant form (ie more work required). We can try to: 1) Factoring the numerator and denominator 2) Cancelling expressions 3) Try direct substitution on the new expression. Note that some factoring techniques you will need to recall: Difference of squares: $a^2 - b^2 = (a-b)(a+b)$ Difference of cubes: $a^3 - b^3 = (a-b)(a^2+ab+b^2)$ Sum of cubes: $a^3 + b^3 = (a+b)(a^2-ab+b^2)$	If the expression is 0/0 and you can factor the expressions in the numerator and denominator.	If the numerator and denominator are both zero, it is possible that a factor $(x - a)$ can appear in both numerator and denominator. Cancelling this factor will allow for the remaining expression to use direct substitution. Note that even though f(a) will be undefined, the value of the limit does not care about x=a it only cares about x approaching a.

Examples: Solving Limits Using Factoring

Example 2:

Determine the limits below:

a) $\lim\limits_{x \to 1} \frac{x-1}{x^2-1}$

b) $\lim\limits_{x \to 3} \frac{x-3}{x^2-5x+6}$

c) $\lim\limits_{x \to -2} \frac{x^3+8}{x^2-4}$

Solution:

a) $\lim\limits_{x \to 1} \frac{x-1}{x^2-1}$

Sub in $x = 1$ gives $\frac{0}{0}$ (indeterminant)

Thus we try factoring:

$$\lim_{x \to 1} \frac{x-1}{x^2-1} = \lim_{x \to 1} \frac{x-1}{(x-1)(x+1)}$$
$$= \lim_{x \to 1} \frac{1}{(x+1)}$$

Since we cancelled a term of the form $x - 1$, our new expression may be evaluated using direct substitution:

$$= \frac{1}{1+1} = \frac{1}{2}$$

b) $\lim\limits_{x \to 3} \frac{x-3}{x^2-5x+6}$

Sub in $x = 3$ gives $\frac{0}{0}$ (indeterminant).

Thus we try factoring:

$$\lim_{x \to 3} \frac{x-3}{x^2-5x+6} = \lim_{x \to 3} \frac{x-3}{(x-3)(x-2)}$$
$$= \lim_{x \to 3} \frac{1}{(x-2)}$$

Since we cancelled a term of the form $x - 3$, our new expression may be evaluated using direct substitution:

$$= \frac{1}{3-2} = 1$$

c) $\lim\limits_{x \to -2} \frac{x^3+8}{x^2-4}$

Sub in $x = -2$ gives $\frac{0}{0}$ (indeterminant).

Thus we try factoring:

$$\lim_{x \to -2} \frac{x^3+8}{x^2-4} = \lim_{x \to -2} \frac{(x+2)(x^2-2x+4)}{(x-2)(x+2)}$$
$$= \lim_{x \to -2} \frac{x^2-2x+4}{(x-2)}$$

Since we cancelled a term of the form $x - (-2)$, our new expression may be evaluated using direct substitution:

$$= \frac{(-2)^2-2(-2)+4}{-2-2} = -3$$

Strategy: Finding Limits using Conjugates

A conjugate occurs when you start with a sum (or difference) of two expressions and change it to a difference (or sum) instead.

For example:

$a+b$	has conjugate	$a-b$
$a-b$	has conjugate	$a+b$
x^2-1	has conjugate	x^2+1
$\sqrt{x}+7$	has conjugate	$\sqrt{x}-7$
$2\sqrt{3-x}+\sqrt{x^2+3}$	has conjugate	$2\sqrt{3-x}-\sqrt{x^2+3}$

How To Use it:

When you try direct substitution and the result is $\frac{0}{0}$ we call this an indeterminant form (ie more work required)

We can try to:

1) **Take the conjugate of any root expressions that are present.**
2) **Multiply the numerator and denominator by the conjugate.**
3) **Only "Expand" the pieces that make a difference of squares.**
4) **Try to simplify the expression. You may need to factor the expression to get out of $\frac{0}{0}$.**

When To Use it:

If the expression is 0/0 and your expression has square roots in the numerator or denominator.

Why this works?

When you multiply a root expression $\sqrt{x}-a$ by its conjugate $\sqrt{x}+a$ you get a difference of squares which simplifies to $x-a^2$. This eliminates the square root allowing you to simplify the expression.

Also, multiplying and dividing by the same expression is multiplying by 1 which does not change the expression.

Examples: Solving Limits Using Conjugates

Example 3: Determine the limits below:

a) $\lim\limits_{x\to 1} \frac{\sqrt{2-x}-1}{x-1}$

Solution:

a) $\lim\limits_{x\to 1} \frac{\sqrt{2-x}-1}{x-1}$

Sub in $x = 1$ gives $\frac{0}{0}$ (indeterminant). Thus we try factoring using the conjugate as we see a root.

$$\lim_{x\to 1} \frac{\sqrt{2-x}-1}{x-1} = \lim_{x\to 1} \frac{\sqrt{2-x}-1}{x-1} \frac{\left(\sqrt{2-x}+1\right)}{\left(\sqrt{2-x}+1\right)}$$

$$= \lim_{x\to 1} \frac{2-x-1}{(x-1)\left(\sqrt{2-x}+1\right)}$$

$$= \lim_{x\to 1} \frac{-x+1}{(x-1)\left(\sqrt{2-x}+1\right)}$$

$$= \lim_{x\to 1} \frac{-1(x-1)}{(x-1)\left(\sqrt{2-x}+1\right)}$$

$$= \lim_{x\to 1} \frac{-1}{\left(\sqrt{2-x}+1\right)}$$

Since we cancelled a term of the form $x - 1$, our new expression may be evaluated using direct substitution:

$$= \frac{-1}{\left(\sqrt{2-1}+1\right)} = -\frac{1}{2}$$

Examples: Solving Limits Using Conjugates

Example 3 continued: Determine the limits below:

b) $\lim\limits_{x\to 2} \frac{\sqrt{2}-\sqrt{x}}{\sqrt{x-1}-1}$

Solution:

b) $\lim\limits_{x\to 2} \frac{\sqrt{2}-\sqrt{x}}{\sqrt{x-1}-1}$

Subbing in $x = 2$ gives $\frac{0}{0}$ which is indeterminant. Thus we try factoring using the conjugate as we see a root.

$$\begin{aligned}
\lim_{x\to 2} \frac{\sqrt{2}-\sqrt{x}}{\sqrt{x-1}-1} &= \lim_{x\to 2} \frac{\sqrt{2}-\sqrt{x}}{\sqrt{x-1}-1} \left(\frac{\sqrt{2}+\sqrt{x}}{\sqrt{2}+\sqrt{x}}\right)\left(\frac{\sqrt{x-1}+1}{\sqrt{x-1}+1}\right) \\
&= \lim_{x\to 2} \frac{2-x}{x-1-1}\left(\frac{\sqrt{x-1}+1}{\sqrt{2}+\sqrt{x}}\right) \\
&= \lim_{x\to 2} \frac{2-x}{x-2}\left(\frac{\sqrt{x-1}+1}{\sqrt{2}+\sqrt{x}}\right) \\
&= \lim_{x\to 2} \frac{-1(x-2)}{x-2}\left(\frac{\sqrt{x-1}+1}{\sqrt{2}+\sqrt{x}}\right) \\
&= \lim_{x\to 2} -\left(\frac{\sqrt{x-1}+1}{\sqrt{2}+\sqrt{x}}\right)
\end{aligned}$$

Since we cancelled a term of the form $x - 2$, our new expression may be evaluated using direct substitution:

$$= -\left(\frac{\sqrt{2-1}+1}{\sqrt{2}+\sqrt{2}}\right) = -\frac{2}{2\sqrt{2}} = -\frac{\sqrt{2}}{2}$$

Strategy: Using Right Side and Left Side Limits

How To Use it:	When To Use it:	Why this works?
When dealing with functions that end or change functions at the point of the limit, we can try to: 1) Take the left side limit. 2) Take the right side limit. 3) See if both limits go to the same value, this will be the value of the limit. Note that for piecewise functions, you will need to know the correct piece to substitute. For absolute value functions, you may need to change them to piecewise functions.	This strategy is used when: 1) The function ends at the limit value ($\sqrt{x}$ ends at 0, $\log(x-2)$ ends at 2, etc…) 2) If the function would change abruptly (piecewise functions and absolute value functions)	Functions that end do not have limits on one side, so we need to consider this as a separate case. Functions that change abruptly (piecewise and absolute value functions) will also greatly affect the limit and need to be handled separately.

Examples: Using Right Side and Left Side Limits

Example 4

Determine the limits below:

a) $\lim_{x\to 1} f(x)$

b) $\lim_{x\to 5} \frac{\sqrt{x-5}}{x}$

c) $\lim_{x\to -2} \frac{x^2-4}{|x+2|}$

Where $f(x) = \begin{cases} x^2+1 & x \geq 1 \\ -x+2 & x < 1 \end{cases}$

Solution:

a) Since the function changes at $x = 1$ and this is the value of our limit, we will need to see what happens on both sides of 1:

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x^2+1$$
$$= (1)^2+1 = 2$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} -x+2$$
$$= -1+2 = 1$$

Since the limits do not meet at the same value, we say that the limit does not exist.

b) This limit is a bit tricky, since at first glance, we would think that the limit goes to 0/5 = 0. However $\sqrt{x-5}$ ends at $x = 5$ and thus there is no limit on the left of $x = 5$.

Thus this limit does not exist.

Examples: Using Right Side and Left Side Limits

Example 4 continued:

Determine the limits below:

a) $\lim_{x \to 1} f(x)$

b) $\lim_{x \to 5} \frac{\sqrt{x-5}}{x}$

c) $\lim_{x \to -2} \frac{x^2-4}{|x+2|}$

Where $f(x) = \begin{cases} x^2+1 & x \geq 1 \\ -x+2 & x < 1 \end{cases}$

Solution:

c) Substituting $x = -2$ gives us 0/0, but to simplify the expression, we would want to eliminate the absolute values in the function.

$$|x+2| = \begin{cases} x+2 & x \geq -2 \\ -(x+2) & x < -2 \end{cases}$$

This means that we must look at both sides of the limit:

$$\begin{aligned} \lim_{x \to -2^+} f(x) &= \lim_{x \to -2^+} \frac{x^2-4}{x+2} \\ &= \lim_{x \to -2^+} \frac{(x+2)(x-2)}{x+2} \\ &= \lim_{x \to -2^+} x-2 \\ &= -2-2 = -4 \end{aligned}$$

$$\begin{aligned} \lim_{x \to -2^-} f(x) &= \lim_{x \to -2^-} \frac{x^2-4}{-(x+2)} \\ &= \lim_{x \to -2^-} -\frac{(x+2)(x-2)}{(x+2)} \\ &= \lim_{x \to -2^-} -x+2 \\ &= 2+2 = 4 \end{aligned}$$

Since the limits do not meet at the same value, we say that the limit does not exist.

Strategy: Trigonometric Limits and Substitution

We will take for granted that we can prove:

$$\lim_{x\to 0}\frac{\sin(x)}{x}=1 \qquad \lim_{x\to 0}\frac{x}{\sin(x)}=1 \qquad \lim_{x\to 0}\frac{1-\cos(x)}{x}=0 \qquad \lim_{x\to 0}\frac{x}{1-\cos(x)}=DNE$$

We will illustrate substitution using these limits.

How To Use it:

Check that the function is indeterminant $(0/0)$ first, then you can:

1) **Make sure that the argument under the limit is going to 0. If not, do a substitution that allows for the argument to go to 0.**
2) **Make the expression have pieces that look like $\frac{\sin(\square)}{\square}$ or $\frac{\square}{\sin(\square)}$ or $\frac{1-\cos(\square)}{\square}$ or $\frac{\square}{1-\cos(\square)}$.**
3) **To do step one, you may need to multiply "1" in creative ways such as introducing $\frac{2}{2}$ or $\frac{x}{x}$. You may also need to "pull constants" out of the limit.**
4) **Split the limits into smaller limits that can be solved.**
5) **Substitute $\square=\theta$ and determine what $\theta\rightarrow$ when you let $\square\rightarrow a$. (If needed)**
6) **Use the identity limits to solve the problem**

When To Use it:

This strategy is used when we are indeterminant $(\frac{0}{0})$ and when there are trig functions involved.

Why this works?

To see a proof for the identity limits, click here.

Substitution is simply a method of changing variables to allow for us to use the "identity" limits.

Examples: Trigonometric Limits and Substitution

Example 5

Determine the limits below:

a) $\lim_{x \to 0} \frac{\sin(x)}{2x}$ b) $\lim_{x \to 0} \frac{x}{\sin(2x)}$ c) $\lim_{x \to 0} \frac{3x^2 \cos(x)}{\sin^2(2x)}$ d) $\lim_{x \to \pi} \frac{(1-\cos(x-\pi))}{\sin(x)}$

Solution:

a) When we sub in $x = 0$ we see we get $\frac{0}{0}$ thus more work is required. We almost have $\frac{\sin(\square)}{\square}$, so pulling out the 2 in the denominator gives us:

$$\begin{aligned}
\lim_{x \to 0} \frac{\sin(x)}{2x} &= \frac{1}{2} \lim_{x \to 0} \frac{\sin(x)}{x} \\
&= \frac{1}{2}(1) \quad \text{(our identity limit)} \\
&= \frac{1}{2}
\end{aligned}$$

b) When we sub in $x = 0$ we see we get $\frac{0}{0}$ thus more work is required. We almost have $\frac{\square}{\sin(\square)}$, we manipulate the limit:

$$\begin{aligned}
\lim_{x \to 0} \frac{x}{\sin(2x)} &= \lim_{x \to 0} \frac{2x}{2\sin(2x)} \\
&= \frac{1}{2} \lim_{x \to 0} \frac{2x}{\sin(2x)}
\end{aligned}$$

Next, we want the argument to be simply θ so we let :

$\theta = 2x$ As $x \to 0$ we get $\theta \to 2(0) = 0$

We substitute this into our limit and get:

$$\begin{aligned}
&= \frac{1}{2} \lim_{\theta \to 0} \frac{\theta}{\sin(\theta)} \\
&= \frac{1}{2}(1) \quad \text{(Our identity Limit)} \\
&= \frac{1}{2}
\end{aligned}$$

Examples: Trigonometric Limits and Substitution

Example 5 continued

Determine the limits below:

a) $\lim_{x \to 0} \frac{\sin(x)}{2x}$ b) $\lim_{x \to 0} \frac{x}{\sin(2x)}$ c) $\lim_{x \to 0} \frac{3x^2 \cos(x)}{\sin^2(2x)}$ d) $\lim_{x \to \pi} \frac{(1-\cos(x-\pi))}{\sin(x)}$

Solution:

c) $\lim_{x \to 0} \frac{3x^2 \cos(x)}{\sin^2(2x)}$

$$= 3 \lim_{x \to 0} \left(\frac{x}{\sin(2x)}\right)\left(\frac{x}{\sin(2x)}\right)(\cos(x))$$

$$= 3 \lim_{x \to 0} \left(\frac{2x}{2\sin(2x)}\right)\left(\frac{2x}{2\sin(2x)}\right)(\cos(x))$$

$$= \frac{3}{4} \lim_{x \to 0} \left(\frac{2x}{\sin(2x)}\right)\left(\frac{2x}{\sin(2x)}\right)(\cos(x))$$

$$= \frac{3}{4} \lim_{x \to 0} \left(\frac{2x}{\sin(2x)}\right) \lim_{x \to 0} \left(\frac{2x}{\sin(2x)}\right) \lim_{x \to 0}(\cos(x))$$

$$= \frac{3}{4} \lim_{x \to 0} \left(\frac{2x}{\sin(2x)}\right) \lim_{x \to 0} \left(\frac{2x}{\sin(2x)}\right)(1)$$

Next, we want the argument to be simply θ so we let :

$\theta = 2x$ As $x \to 0$ we get $\theta \to 2(0) = 0$

We substitute this into our limit and get:

$$= \frac{3}{4}\left(\lim_{\theta \to 0} \frac{\theta}{\sin(\theta)}\right)\left(\lim_{\theta \to 0} \frac{\theta}{\sin(\theta)}\right)$$

$$= \frac{3}{4}(1)(1) = \frac{3}{4}$$ (Our identity Limit)

d) We first note that $x \to \pi$ is appearing rather than $x \to 0$, so we need to fix the argument of the limit first. We have $x \to \pi$ means $x - \pi \to 0$ so we let $\theta = x - \pi$ (or $x = \theta + \pi$) to get $\theta \to \pi - \pi = 0$

$$\lim_{x \to \pi} \frac{(1-\cos(x-\pi))}{\sin(x)} = \lim_{\theta \to 0} \frac{(1-\cos(\theta))}{\sin(\theta+\pi)}$$

$$= \lim_{\theta \to 0} \frac{(1-\cos(\theta))}{\sin(\theta)\cos(\pi)+\cos(\theta)\sin(\pi)}$$ (Sin(A+B) identity)

$$= \lim_{\theta \to 0} \frac{(1-\cos(\theta))}{-\sin(\theta)+(0)\cos(\theta)}$$

$$= \lim_{\theta \to 0} \frac{(1-\cos(\theta))}{-\sin(\theta)}$$

$$= -\lim_{\theta \to 0} \frac{(1-\cos(\theta))}{\sin(\theta)}$$

$$= -\lim_{\theta \to 0} \frac{(1-\cos(\theta))}{\sin(\theta)}\left(\frac{\theta}{\theta}\right)$$

$$= -\lim_{\theta \to 0} \frac{(1-\cos(\theta))}{\theta}\left(\frac{\theta}{\sin(\theta)}\right)$$

$$= -\lim_{\theta \to 0} \frac{(1-\cos(\theta))}{\theta} \lim_{\theta \to 0}\left(\frac{\theta}{\sin(\theta)}\right)$$

$$= -(0)(1) = 0$$ (Identity Limits)

Graphical Definition of Continuity

If a function can be drawn over an interval without the need to lift the pencil off of the page, we say the function is <u>continuous</u> on that interval. If at any point the pencil must be lifted, we say the function is <u>discontinuous</u> (at that point)

<u>Main ways a function can be discontinuous:</u>

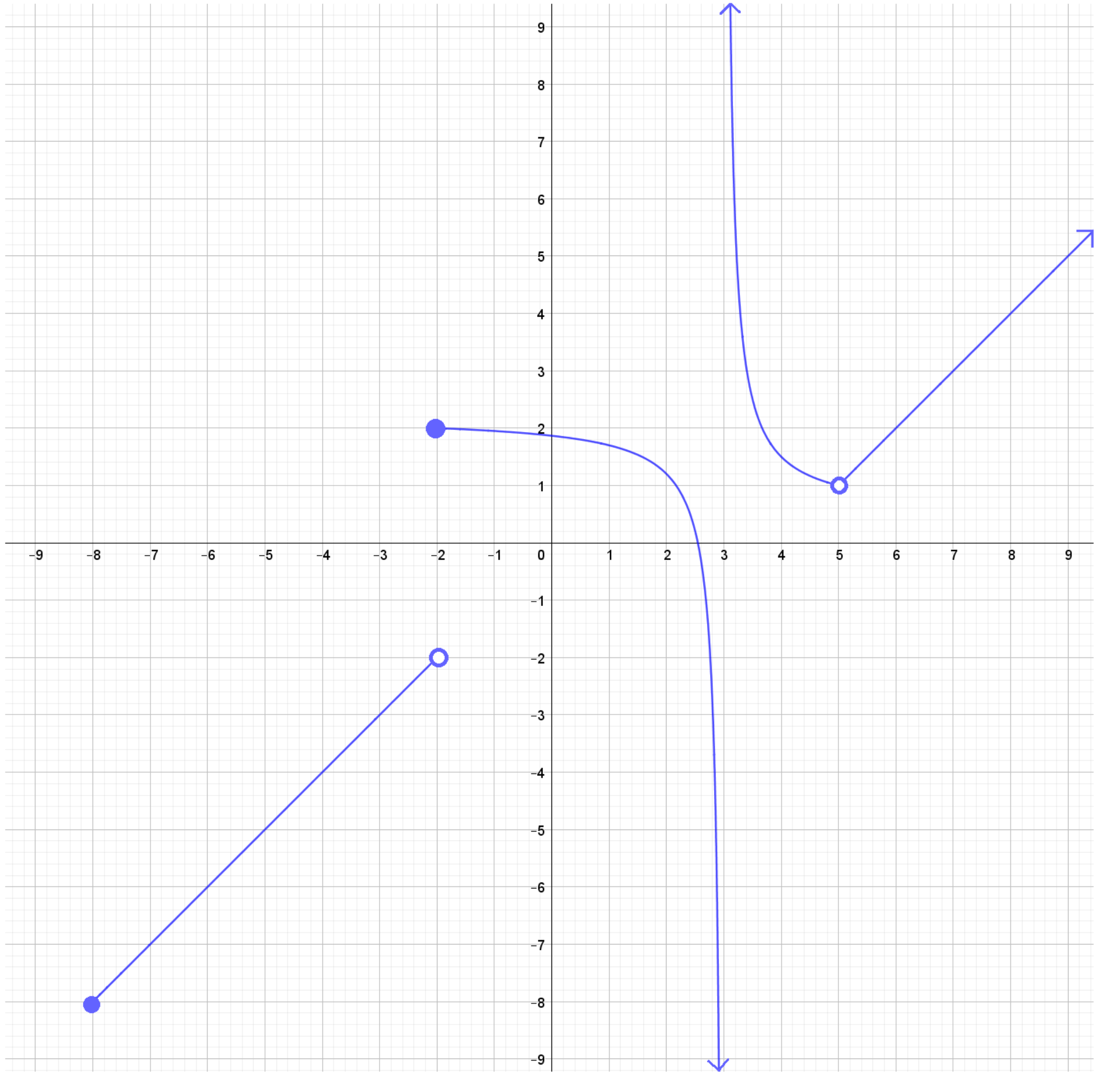
1) There is a hole in the interval (replaceable discontinuity)
2) There is a jump in the function somewhere in the interval (jump discontinuity)
3) There is a vertical asymptote in the interval (asymptote discontinuity)
4) The function ends abruptly (endpoint discontinuity)

We note that a function is continuous at $x = a$ when $f(a) = \lim_{x \to a} f(x)$ (That is when the function value is the same as the limit provided that both of these things exist!)

Examples: Identifying Continuity Using Graphs

Example 5:

Consider the graph below. Determine the interval where the function is continuous. At any point where a function is discontinuous, identify the type of discontinuity.



<u>Solution:</u>

We have an endpoint discontinuity at $x = -8$

We have a jump discontinuity at $x = -2$

We have an asymptote discontinuity at $x = 3$

We have a replaceable discontinuity at $x = 5$

Thus our function is continuous on:

$(-8, -2) \cup (-2,3) \cup (3,5) \cup (5, \infty)$

Again, we note that at any point where the function is continuous:

$$\lim_{x \to a} f(x) = f(a)$$

Definition of Continuity

A function $f(x)$ is continuous at a point $x = a$ if all 3 of the following conditions are satisfied:

1. $f(a)$ is defined.
2. $\lim_{x \to a} f(x)$ exists.
3. $\lim_{x \to a} f(x) = f(a)$

A function $f(x)$ is continuous on an interval means that $f(x)$ is continuous at every point in the interval.

If f and g are continuous at $x = a$, then

1. $(f(x))^r$ is continuous at $x = a$, whenever it is defined. ($r \in$)
2. $f \pm g$ is continuous at $x = a$.
3. fg is continuous at $x = a$.
4. $\frac{f}{g}$ is continuous at $x = a$, provided $g(a) \neq 0$.
5. A polynomial function is continuous at all $x \in \Re$.
6. The rational function $R(x) = \frac{p(x)}{q(x)}$ is continuous at all $x \in \Re$ where $q(x) \neq 0$

Strategy: Testing Continuity Algebraically

How To Use it:	When To Use it:	Why this works?
To identify key points to test at a function we look at places where we think the function may be discontinuous: 1) Divisions by zeros 2) Log(0) 3) Tan(x) that produce asymptotes 4) Root functions that end. 5) Piecewise functions where the function changes. Note you must also make sure that each piece is continuous on its domain as well! To test a particular x −value we: 1) Determine $f(a)$ 2) Determine $\lim_{x\to a} f(x)$ (note we may need to check both sides of the limit) 3) Make sure that all values exist and are equal to each other.	When determining if and when a function is continuous at a point.	The definition of continuity requires that $\lim_{x\to a} f(x) = f(a)$. This strategy is simply using the definition.

Examples: Continuity

Example 1:

Determine where the function $f(x) = \frac{1}{x^2-4}$ is continuous.

Solution:

We note that the domain of the function can be found by factoring the denominator:

$$f(x) = \frac{1}{(x+2)(x-2)}$$

Since we cannot have a denominator of 0, we get that this function is discontinuous at $x = \pm 2$ (there is no point here so $f(a)$ does not exist). This means that the function is continuous on $(-\infty, -2) \cup (-2,2) \cup (2, \infty)$

Examples: Continuity

Example 2:

Determine where the function is continuous: $f(x) = \begin{cases} \sqrt{x+2} & x \leq 2 \\ x^2 - 2 & 2 < x < 5 \\ 2^x & x \geq 5 \end{cases}$

Solution:

We first note that each piece is continuous except the root function which has an endpoint discontinuity. We can determine where the function ends as a root function ends at $\sqrt{0}$ thus we solve for the endpoint: $x + 2 = 0$ gives $x = -2$. Thus the function is discontinuous at $x \leq -2$ (it is not continuous at $x = -2$ as there is no limit on the left side of the root function).

The other places of discontinuity are on sudden changes in the function. We test them using left side and right side limits:

$\underline{x = 2}$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \sqrt{x+2}$$

$$= \sqrt{2+2} \quad \text{(Root functions are continuous above 0)}$$

$$= 2$$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} x^2 - 2$$

$$= 2^2 - 2 \quad \text{(quadratics are continuous everywhere)}$$

$$= 2$$

$$f(2) = \sqrt{2+2} = 2$$

Thus since we have all values exist and are all equal, the function is continuous at $x = 2$

Examples: Continuity

Example 2 Continued:

Determine where the function is continuous: $f(x) = \begin{cases} \sqrt{x+2} & x \leq 2 \\ x^2 - 2 & 2 < x < 5 \\ 2^x & x \geq 5 \end{cases}$

Solution:

$x = 5$

$$\lim_{x \to 5^-} f(x) = \lim_{x \to 5^-} x^2 - 2$$
$$= 5^2 - 2 \quad \text{(quadratics are continuous everywhere)}$$
$$= 23$$

$$\lim_{x \to 5^+} f(x) = \lim_{x \to 5^+} 2^x$$
$$= 2^5 \quad \text{(exponentials are continuous everywhere)}$$
$$= 32$$

$f(5) = 2^5 = 32$

Thus since we have all values exist, but they are not all equal. Thus the function is **not** continuous at $x = 5$

$\therefore$ If we place all pieces together we get that the function is continuous on $(-2,5) \cup (5, \infty)$

Examples: Continuity

Example 3:

Determine the value of a that would allow the function to be continuous on $(-\infty, \infty)$: $f(x) = \begin{cases} x^2 + ax & x \leq 1 \\ -x + 2a & x > 1 \end{cases}$

Solution:

We first note that each piece is continuous (quadratics and lines are both continuous everywhere). However, there may be a jump at $x = 1$. So we require to check this point more carefully:

$\underline{x = 1}$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^2 + ax$$
$$= 1^2 + a(1) \quad \text{(quadratics are continuous everywhere)}$$
$$= 1 + a$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} -x + 2a$$
$$= -1 + 2a \quad \text{(linear functions are continuous everywhere)}$$

Since we require both of these limits to be equal as we desire continuity, we can set them equal to each other to solve for a:

$$1 + a = -1 + 2a$$
$$2 = a$$

We see that if $f(x) = \begin{cases} x^2 + 2x & x \leq 1 \\ -x + 4 & x > 1 \end{cases}$ then we have $\lim_{x \to 1^-} f(x) = 1 + 2 = 3 = -1 + 4 = \lim_{x \to 1^+} f(x)$ and we also note that $f(1) = 1 + 2 = 3$ thus all the values exist and are equal.

$\therefore a = 2$ will allow for the function to be continuous on the entire domain $(-\infty, \infty)$.

Strategy: Intermediate Value Theorem

Intermediate Value Theorem:

Given:

1. $f(x)$ **is a continuous on a closed interval** $[a, b]$

2. y **is any number that is in between** $f(a)$ **and** $f(b)$

Conclusion:We will always be able to find some c **in the interval** $[a, b]$ **such that** $f(c) = y$

We typically use the intermediate value theorem to determine if a function has roots in a given interval.
If $f(x)$ **is a continuous on a closed interval** $[a, b]$ **and** $f(a)$ **and** $f(b)$ **have opposite signs, then there is at least one solution of the equation** $f(x)$ **=0 in the interval** $[a, b]$ **.**

How To Use it:

1) **Ensure (state) that you have a continuous function on** $[a, b]$
2) **Ensure that** y **(usually** $y = 0$**) is in between** $f(a)$ **and** $f(b)$**.**
3) **Conclude that we have a** c **in the interval of** $[a, b]$ **such that** $f(c) = y$**.**

When To Use it:

Typically we use this for finding roots. But we can use it to say that we can always find an input c that will produce $f(c) = y$ as long as y is in between $f(a)$ and $f(b)$.

Why this works?

Try picking two points (say $(1,2)$ and $(10,15)$. Now connect these two points continuously (that is do not jump, use an asymptote, etc…). Be sure that it is a function (passes the vertical line test).

You should notice that this can only be done if we hit every possible y value in between 2 and 15 (you may hit others,

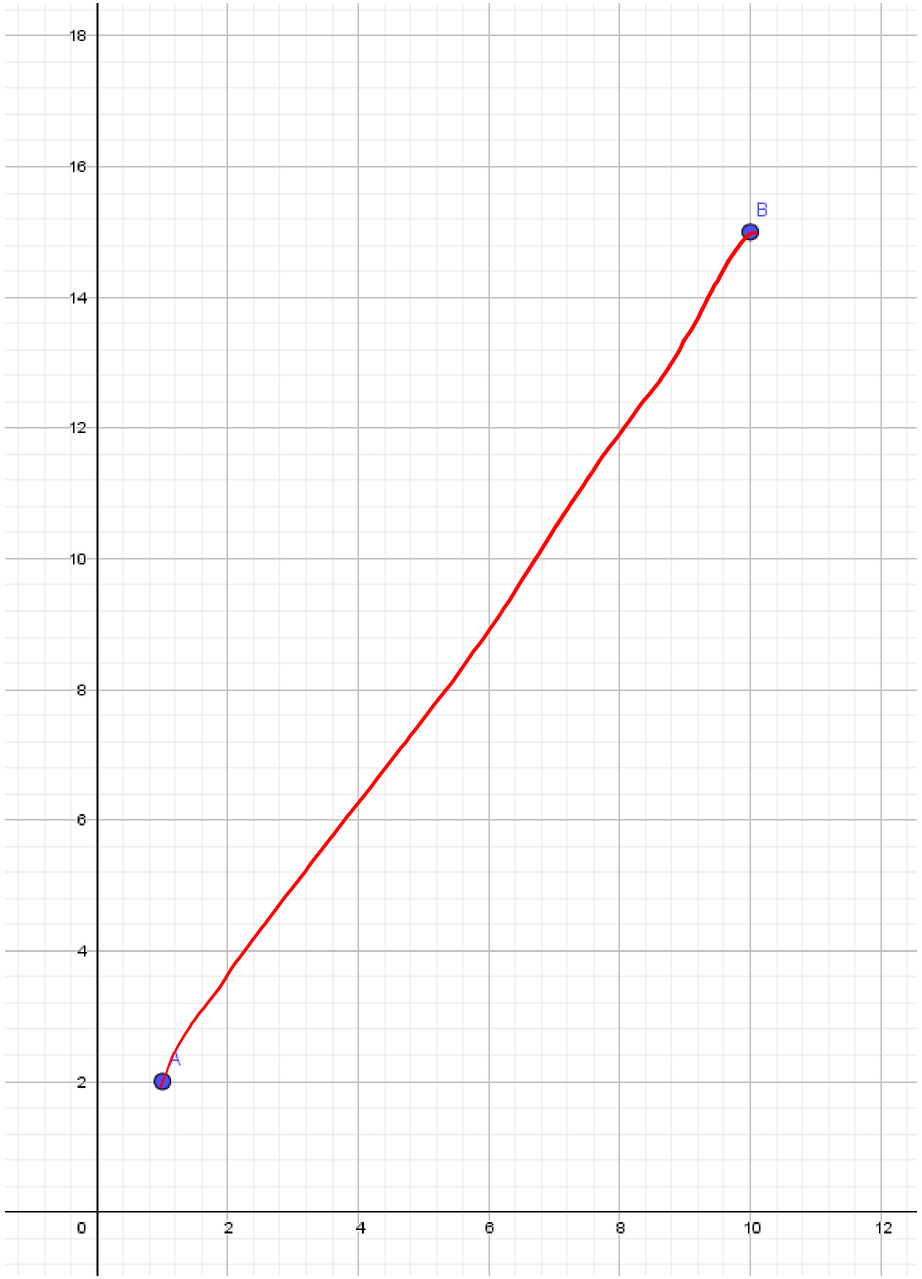
Examples: Intermediate Value Theorem

Example 4:

Draw a continuous function from (1, 2) to (10,15) in such a way that:

a) There are no y values less than 2 or greater than 15.

Solution:



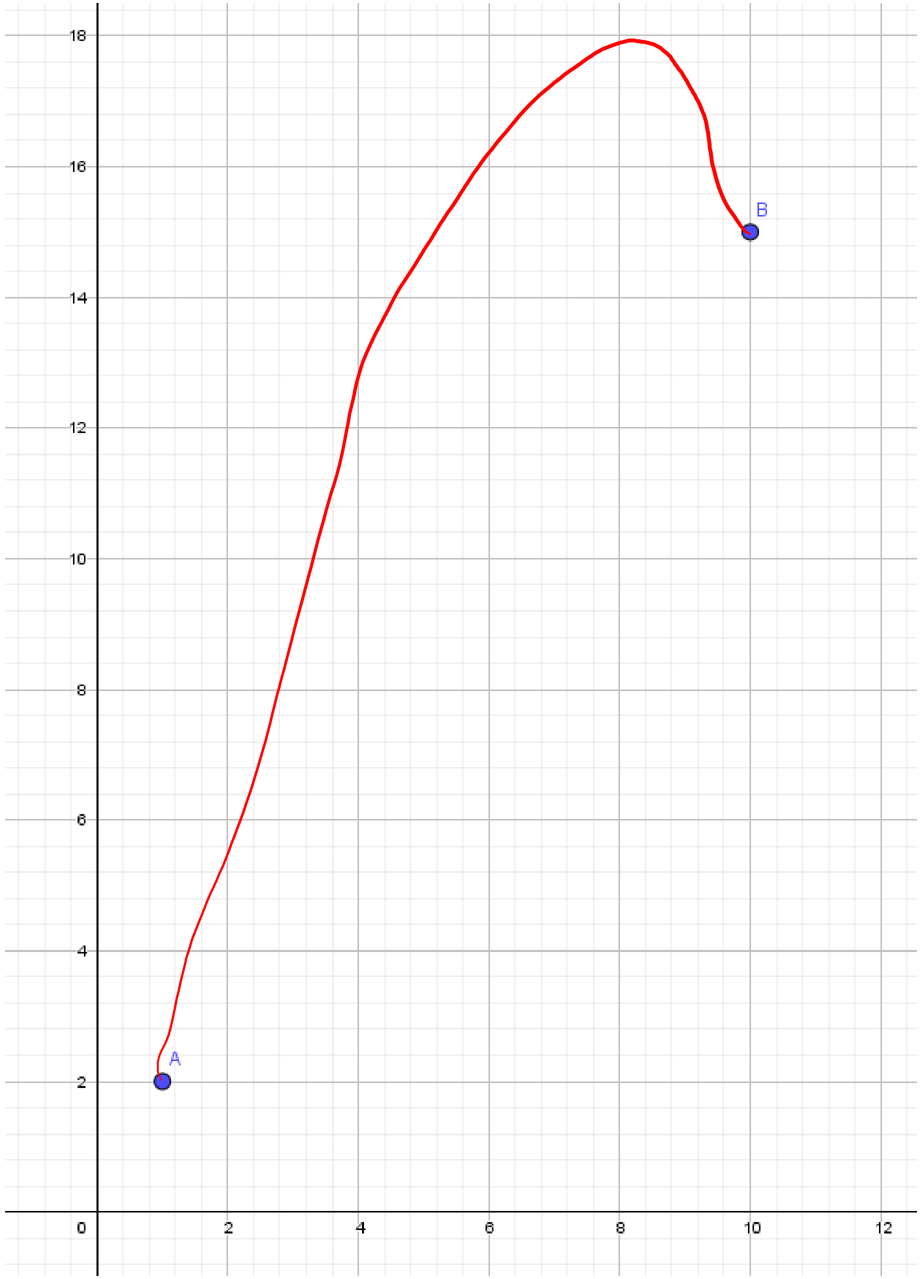
Examples: Intermediate Value Theorem

Example 4:

Draw a continuous function from (1, 2) to (10,15) in such a way that:

b) There are no y values less than 2 but some values greater than 15.

Solution:



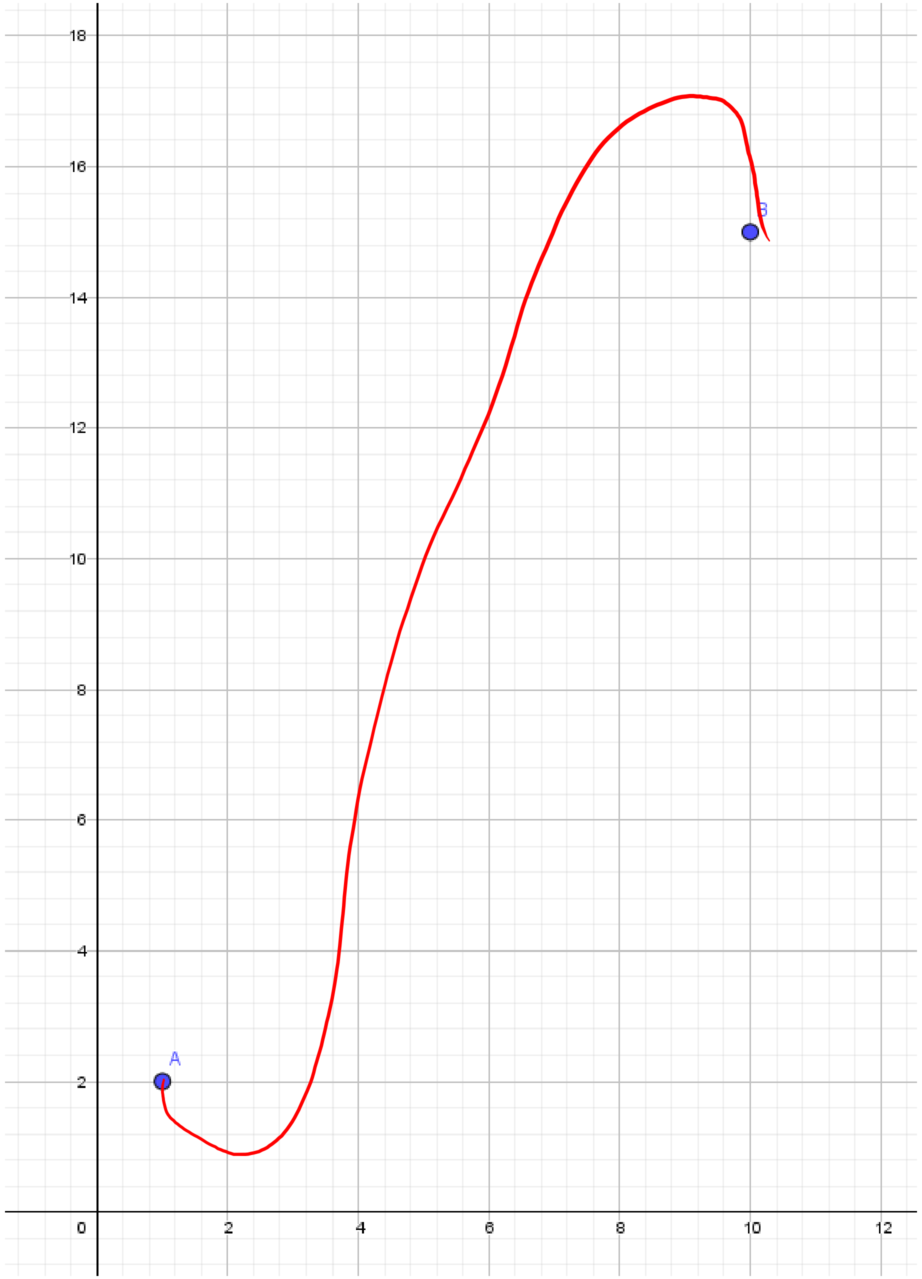
Examples: Intermediate Value Theorem

Example 4:

Draw a continuous function from (1, 2) to (10,15) in such a way that:

c) There are y values less than 2 and some that are greater than 15.

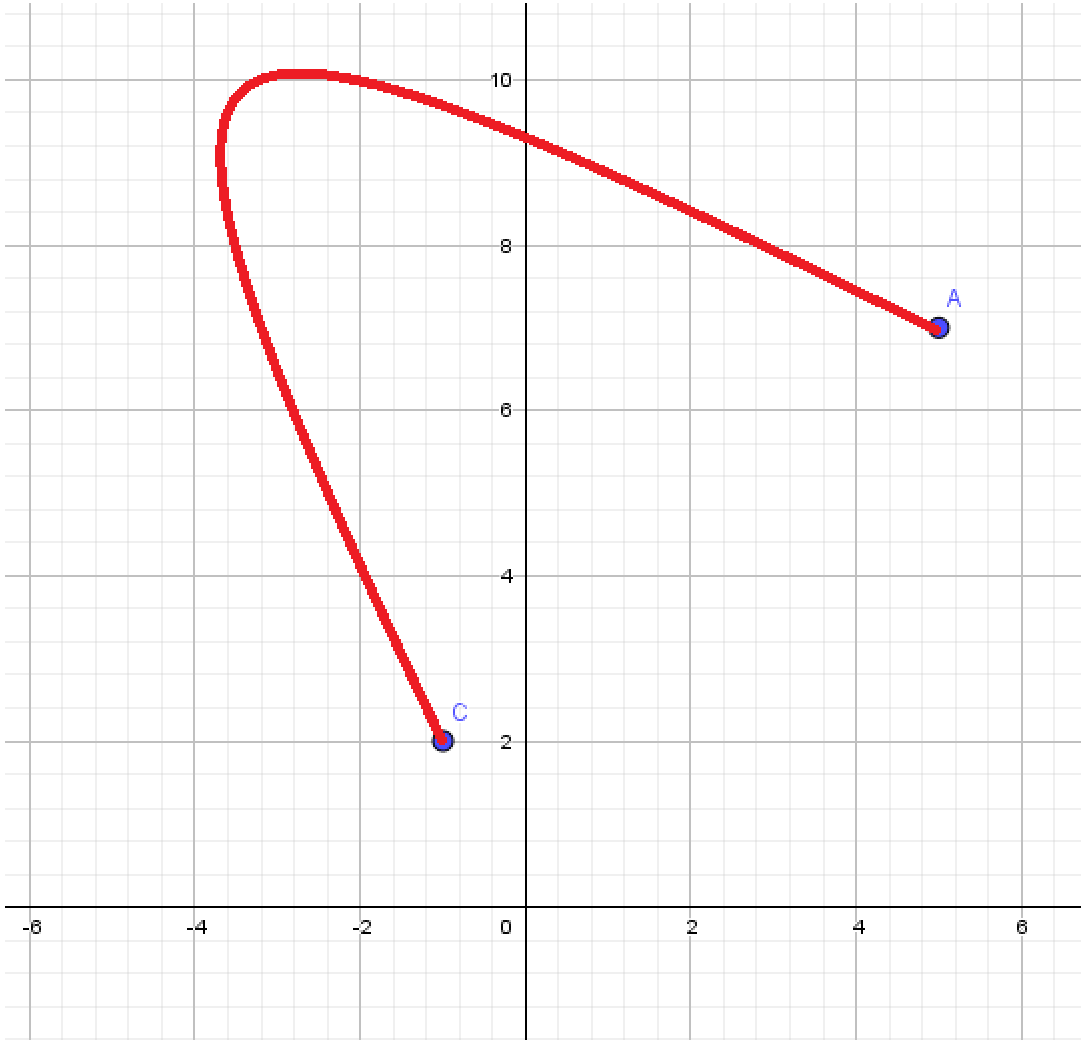
Solution:



Examples: Intermediate Value Theorem

Example 5:

Note that there are no y values between $f(-1) = 2$ and $f(5) = 7$ for $x \in [-1,5]$. What condition(s) of the intermediate value theorem is not present?



Solution:

1) The graph is continuous

2) The graph is **not** a function

3) We are considering a closed interval $[-1,5]$

Examples: Intermediate Value Theorem

Example 6:

Show that $x^3 - 5x^2 + x = -2$ somewhere on the interval $[0,3]$.

Solution:

1) We note that $x^3 - 5x^2 + x$ is a continuous function everywhere.

2) Next we check the endpoints:

$f(0) = 0^3 - 5(0)^2 + 0 = 0$

$f(3) = 3^3 - 5(3)^2 + 3 = -15$

Since -2 is in between -15 and 0 and the function is continuous, by the intermediate value theorem there must be some $y \in [0,3]$ where $f(y) = -2$.

This means there is at least one solution to $x^3 - 5x^2 + x = -2$ somewhere on the interval $[0,3]$.

Examples: Intermediate Value Theorem

Example 7:

Let $f(x) = 2\cos(x) - 2^x$. Show that there must be a zero on the interval $[0,2]$

Solution:

1) We note that 2 cos(x) and 2^x are both continuous everywhere, so the sum of these are also continuous. Thus the function is continuous on all x.

2) If we check the endpoints we see:

$f(0) = 2\cos(0) - 2^0 = 1$

$f(2) = 2\cos(2) - 2^2 = k - 4$ (we have no idea without a calculator what the value of $2\cos(2)$ is)

Although without a calculator, we are unsure what k is, we know that $k - 4 < 0$ since $-2 \leq 2\cos(x) \leq 2$ which means this function would never get higher than 4. Thus we subtract 4, we must be a value that is negative

This means we know that f is continuous and 0 must be in between $f(0)$ and $f(2)$ (as $k - 4$ is negative and 1 is positive), which means that (by the intermediate value theorem) there must be some $c \in [0,2]$ where $f(c) = 0$.

This means there is at least one zero in between 0 and 2.

Strategy: Calculating Limits involving $\frac{k}{0}$ when $k \neq 0$.

How To Use it:	When To Use it:	Why this works?
If we have $\frac{k}{0}$ then we must: 1) Split the limit into left and right limits (if needed). 2) In each limit, identify if the zero is "slightly positive" or "slightly negative". 3) Evaluate the limit to either ∞ or $-\infty$ depending on the sign.	When we have a limit where $\frac{k}{0}$ when $k \neq 0$.	When we divide by a fraction, we are actually multiplying by the reciprocal. So as we get closer and closer to 0 (say $\frac{1}{10}, \frac{1}{100}, \frac{1}{1000}$, …) we are going to multiply k by a larger and larger amount: ($10, 100, 1000$, …). This means that the value will get really large and positive (∞) or really large and negative ($-\infty$).

Example 1:

Find the value of the limit: $\lim_{x \to 3^+} \frac{5}{3-x}$

Solution:

We note that if we sub in $x = 3$ we will get $\frac{5}{0}$. We only need to look at the right side of 3 since the limit is going to 3^+. We get:

$\lim_{x \to 3^+} \frac{5}{3-x} \to \lim_{x \to 3^+} \frac{5}{3-3^+}$

$\to \lim_{x \to 3^+} \frac{5}{0^-}$ (3 take away a number larger than 3 will be close to 0, but negative)

$\to -\infty$ (positive divided by a negative gives a negative, and it gets infinitely large when dividing 0)

Examples: Limits of $\frac{k}{0}$

Example 2:

Find the value of the limit: $\lim_{x \to 5} \frac{x+1}{x^2-25}$

Solution:

We note that if we sub in $x = 5$ we will get $\frac{6}{0}$. We will need to look at both sides of the limit to see what happens:

$$\lim_{x \to 5^+} \frac{x+1}{x^2-25} \to \lim_{x \to 5^+} \frac{5^+ + 1}{(5^+)^2 - 25}$$

$$\to \lim_{x \to 5^+} \frac{6^+}{25^+ - 25}$$

$$\to \lim_{x \to 5^+} \frac{6^+}{0^+}$$

$$\to \infty$$

$$\lim_{x \to 5^-} \frac{x+1}{x^2-25} \to \lim_{x \to 5^-} \frac{5^- + 1}{(5^-)^2 - 25}$$

$$\to \lim_{x \to 5^-} \frac{6^-}{25^- - 25}$$

$$\to \lim_{x \to 5^-} \frac{6^-}{0^-}$$

$$\to -\infty$$

Since the limits do not match on both sides, we say that the limit does not exist.

Examples: Limits of $\frac{k}{0}$

Example 3:

Find the value of the limit: $\lim_{x \to 0} \frac{x^2 - 1}{\sin^2(x)}$

Solution:

We note that if we sub in $x = 0$ we will get $\frac{-1}{0}$. We will need to look at both sides of the limit to see what happens:

$$\lim_{x \to 0^+} \frac{x^2 - 1}{\sin^2(x)} \quad \to \lim_{x \to 0^+} \frac{\left(0^+\right)^2 - 1}{(\sin(0^+))^2}$$

$$\to \lim_{x \to 0^+} \frac{0^+ - 1}{(0^+)^2}$$

$$\to \lim_{x \to 0^+} \frac{-1^+}{0^+}$$

$$\to -\infty$$

$$\lim_{x \to 0^-} \frac{x^2 - 1}{\sin^2(x)} \quad \to \lim_{x \to 0^-} \frac{(0^-)^2 - 1}{(\sin(0^-))^2}$$

$$\to \lim_{x \to 0^+} \frac{0^+ - 1}{(0^-)^2}$$

$$\to \lim_{x \to 0^+} \frac{-1^+}{0^+}$$

$$\to -\infty$$

$\therefore$ The limit goes to $-\infty$

Strategy: Limits involving $\frac{\infty}{k}$ or $\frac{k}{\infty}$. (Assuming $k \neq 0$)

How To Use it:

When taking a limit to infinity, you may want to treat ∞ as a number. We can do so (to a certain extent), but keep in mind that. If the expressions is of the form:

$\frac{\infty}{k}$: This goes to ∞ or $-\infty$ depending on the sign of k

$\frac{k}{\infty}$: This goes to 0

$\infty + k$: This goes to ∞

$\infty + \infty$: This goes to ∞

$-\infty + k$: This goes to $-\infty$

$-\infty - \infty$: This goes to $-\infty$

For other functions, look at the graph and determine what happens as $x \to \infty$ or as $x \to -\infty$

Important note:

$\infty - \infty$ and $\frac{\infty}{\infty}$ are special cases that are indeterminant (require more work) we will explore these cases separately.

When To Use it:

If a limit simplifies to something of the form of $\frac{k}{\infty}$ or $\frac{\infty}{k}$.

Why this works?

Infinity is the process of growing very large. Thus if we multiply infinity by k the number will still be growing very large (and either stay positive or change to negative depending on k)

If we divide a constant by a number that is approaching infinity, this is similar to answering: "If I have 10 cookies and I have to divide amongst the population of the universe, what does everyone get?" The result is 0.

As for adding/subtracting a constant to infinity, adding any number will never change the fact that the infinity is growing large.

As for combining infinity, if they are of the same sign, it simply grows faster in that direction.

Examples: Limits involving $\frac{\infty}{k}$ or $\frac{k}{\infty}$. (Assuming $k \neq 0$)

Example 4:

Find the value of the limit: $\lim_{x\to\infty} \frac{3x}{1+\frac{1}{x}}$

Solution:

If we "substitute" the idea of infinity, we get:

$$\lim_{x\to\infty} \frac{3x}{1+\frac{1}{x}} = \lim_{x\to\infty} \frac{3\infty}{1+\frac{1}{\infty}}$$

$$= \lim_{x\to\infty} \frac{\infty}{1+0}$$

($k\infty = \infty$ as k and ∞ are both positive, $\frac{1}{\infty} \to 0$)

$$= \lim_{x\to\infty} \frac{\infty}{1}$$

$$= \infty$$

($\frac{\infty}{k} \to \infty$ as k and ∞ are both positive).

Example 5:

Find the value of the limit: $\lim_{x\to-\infty} \frac{e^x+2}{|x|}$

Solution:

If we "substitute" the idea of infinity, we get:

$$\lim_{x\to-\infty} \frac{e^x+2}{|x|} = \lim_{x\to-\infty} \frac{e^{-\infty}+2}{|-\infty|}$$

$$= \lim_{x\to-\infty} \frac{0+2}{|-\infty|}$$

(As $x \to -\infty$ the graph of e^x goes towards the horizontal asymptote of 0)

$$= \lim_{x\to-\infty} \frac{0+2}{\infty}$$

(As $x \to -\infty$ the graph of $|x|$ goes towards ∞)

$$= 0$$

($\frac{k}{\infty}$ always goes to 0 when k is constant)

Strategy: ∞/∞ using the highest term in the denominator

How To Use it:	When To Use it:	Why this works?
Ensure that your expression is expanded and simplified as much as possible. If you have $\frac{\infty}{\infty}$ we can try: 1) Determine the highest term in the denominator 2) Divide all terms in the numerator and denominator and simplify the result to find the limit. Note that you will need to be comfortable with exponent laws to simplify as much as you can.	When using limits that $x \to \infty$ a limit evaluates to $\frac{\infty}{\infty}$ and the terms have been expanded in the question.	Dividing by the highest term in the denominator allows for one term to change to a constant, and the remaining terms to change into 0 in the denominator. If the numerator simplifies nicely, then we can calculate the limit. Since dividing both the numerator and denominator by the same value does not change the fraction, we will get an equivalent limit.

Example 6:

Find the value of the limit: $\lim_{x\to\infty} \frac{5x^2+3x+1}{2x^2-3x+2}$

Solution:

If we "substitute" the idea of infinity, we get $\frac{\infty}{\infty}$. The highest term in the denominator is x^2:

$$\lim_{x\to\infty} \frac{5x^2+3x+1}{2x^2-3x+2} = \lim_{x\to\infty} \frac{5x^2+3x+1}{2x^2-3x+2} \div \frac{x^2}{x^2}$$

$$= \lim_{x\to\infty} \frac{\frac{5x^2}{x^2}+\frac{3x}{x^2}+\frac{1}{x^2}}{\frac{2x^2}{x^2}-\frac{3x}{x^2}+\frac{2}{x^2}}$$

$$= \lim_{x\to\infty} \frac{5+\frac{3}{x}+\frac{1}{x^2}}{2-\frac{3}{x}+\frac{2}{x^2}}$$

$$= \lim_{x\to\infty} \frac{5+0+0}{2-0+0}$$

$$= \frac{5}{2}$$

Examples: ∞/∞ using the highest term in the denominator

Example 7:

Find the value of the limit: $\lim\limits_{x\to\infty} \frac{\sqrt{x^5+x^2+x}-1}{2x^3-1}$

Solution:

If we "substitute" the idea of infinity, we get $\frac{\infty}{\infty}$. The highest term in the denominator is x^3:

$$\lim_{x\to\infty} \frac{\sqrt{x^3+x^2+x}-1}{2x^3-1} = \lim_{x\to\infty} \frac{\sqrt{x^5+x^2+x}-1}{2x^3-1} \div \frac{x^3}{x^3}$$

$$= \lim_{x\to\infty} \frac{\sqrt{\frac{x^5}{x^6}+\frac{x^2}{x^6}+\frac{x}{x^6}}-\frac{1}{x^3}}{\frac{2x^3}{x^3}-\frac{1}{x^3}}$$

(Since $x^3 = x^{\frac{6}{2}} = \sqrt{x^6}$ ∴ bringing x^3 inside the root makes it x^6)

$$= \lim_{x\to\infty} \frac{\sqrt{\frac{1}{x}+\frac{1}{x^4}+\frac{1}{x^5}}-\frac{1}{x^3}}{2-\frac{1}{x^3}}$$

$$= \lim_{x\to\infty} \frac{\sqrt{0+0+0}-0}{2-0}$$

$$= 0$$

Examples: ∞/∞ using the highest term in the denominator

Example 8:

Find the value of the limit: $\lim\limits_{x\to\infty} \frac{\sqrt[3]{2x^2+x^6}-2x^2}{\sqrt{x^4+x^2}-x}$

Solution:

If we “substitute” the idea of infinity, we get $\frac{\infty}{\infty}$. The highest term in the denominator is $\sqrt{x^4} = x^2$:

$$\lim_{x\to\infty} \frac{\sqrt[3]{2x^2+x^6}-2x^2}{\sqrt{x^4+x^2}-x} = \lim_{x\to\infty} \frac{\sqrt[3]{2x^2+x^6}-2x^2}{\sqrt{x^4+x^2}-x} \div \frac{x^2}{x^2}$$

$$= \lim_{x\to\infty} \frac{\sqrt[3]{\frac{2x^2}{x^6}+\frac{x^6}{x^6}}-\frac{2x^2}{x^2}}{\sqrt{\frac{x^4}{x^4}+\frac{x^2}{x^4}}-\frac{x}{x^2}}$$

(Since $x^2 = x^{\frac{6}{3}} = \sqrt[3]{x^6}$ ∴ bringing x^2 inside the cube root makes it x^6)

(Since $x^2 = x^{\frac{4}{2}} = \sqrt[2]{x^4}$ ∴ bringing x^2 inside the square root makes it x^4)

$$= \lim_{x\to\infty} \frac{\sqrt[3]{\frac{2}{x^4}+1}-2}{\sqrt{1+\frac{1}{x^2}}-\frac{1}{x}}$$

$$= \lim_{x\to\infty} \frac{\sqrt[3]{0+1}-2}{\sqrt{1+0}-0}$$

$$= -1$$

Strategy: $\infty - \infty$

How To Use it:	When To Use it:	Why this works?
1) Simplify using log laws (if present) or using the conjugate (if roots are present) to get rid of $\frac{\infty}{\infty}$ 2) This typically results in $\frac{\infty}{\infty}$, and thus the highest power in the denominator can be used. Note: If the expression cannot be simplified, then try factoring the largest exponent to see what happens as $x \to \infty$ and $x \to -\infty$.	If the expression is not a fraction and you want to evaluate $\infty - \infty$	Simplifying/using the conjugate allows us to potentially change the expression into something that we have worked on before (∞/∞).

Example 9:

Find the value of the limit: $\lim_{x\to\infty} \log(2x^2 + x) - \log(3x^2 - 1)$

Solution:

If we "substitute" the idea of infinity, we get $\infty - \infty$.

$$\lim_{x\to\infty} \log(2x^2 + x) - \log(3x^2 - 1) \qquad = \lim_{x\to\infty} \log\left(\frac{2x^2+x}{3x^2-1}\right)$$

Inside the log we now have $\frac{\infty}{\infty}$ which means we can use our highest power in denominator strategy:

$$= \lim_{x\to\infty} \log\left(\frac{2x^2+x}{3x^2-1} \div \frac{x^2}{x^2}\right)$$

$$= \lim_{x\to\infty} \log\left(\frac{\frac{2x^2}{x^2}+\frac{x}{x^2}}{\frac{3x^2}{x^2}-\frac{1}{x^2}}\right)$$

$$= \lim_{x\to\infty} \log\left(\frac{2+\frac{1}{x}}{3-\frac{1}{x^2}}\right)$$

$$= \lim_{x\to\infty} \log\left(\frac{2+0}{3-0}\right)$$

$$= \log\left(\frac{2}{3}\right)$$

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Examples: $\infty - \infty$

Example 10:

Find the value of the limit: $\lim\limits_{x\to\infty} \sqrt{5x^2 - x} - \sqrt{7x^3 + 10x^2}$

Solution:

If we "substitute" the idea of infinity, we get $\infty - \infty$. Using the conjugate we get:

$$\lim_{x\to\infty} \sqrt{5x^2 - x} - \sqrt{7x^3 - x^2} = \lim_{x\to\infty} \sqrt{5x^2 - x} - \sqrt{7x^3 + 10x^2}\left(\frac{\sqrt{5x^2-x}+\sqrt{7x^3+10x^2}}{\sqrt{5x^2-x}+\sqrt{7x^3+10x^2}}\right)$$

$$= \lim_{x\to\infty}\left(\frac{5x^2-x-[7x^3+10x^2]}{\sqrt{5x^2-x}+\sqrt{7x^3+10x^2}}\right)$$ (don't forget the brackets here!)

$$= \lim_{x\to\infty}\left(\frac{-7x^3-5x^2-x}{\sqrt{5x^2-x}+\sqrt{7x^3+10x^2}}\right)$$

Inside the log we now have $\frac{\infty}{\infty}$ as both numerator and denominator contain infinity:

$$= \lim_{x\to\infty}\left(\frac{-7x^3-5x^2-x}{\sqrt{5x^2-x}+\sqrt{7x^3+10x^2}} \div \frac{x^{3/2}}{x^{3/2}}\right)$$

$$= \lim_{x\to\infty}\left(\frac{-7x^{\frac{3}{2}}-5x^{\frac{1}{2}}-x^{-\frac{1}{2}}}{\sqrt{5x^{-1}-x^{-2}}+\sqrt{7+10x^{-1}}}\right)$$

$$= \lim_{x\to\infty}\left(\frac{-7x^{\frac{3}{2}}-5x^{\frac{1}{2}}-\frac{1}{\sqrt{x}}}{\sqrt{\frac{5}{x}-\frac{1}{x^2}}+\sqrt{7+\frac{10}{x}}}\right)$$

$$= \lim_{x\to\infty}\left(\frac{-7(\infty)-5(\infty)-0}{\sqrt{0-0}+\sqrt{7-0}}\right)$$

$$= \lim_{x\to\infty}\left(\frac{-\infty}{\sqrt{7}}\right) \qquad = -\infty$$

Examples: $\infty - \infty$

Example 11:

Find the value of the limit: $\lim_{x\to\infty} \arctan(x^2 - x^4)$

Solution:

If we “substitute” the idea of infinity, we get $\infty - \infty$ inside of an arctan function. We can’t to simplify this, but factoring gives:

$$
\begin{aligned}
\lim_{x\to\infty} \arctan(x^2 - x^4) &= \lim_{x\to\infty} \arctan\left(x^4\left(\frac{1}{x^2} - 1\right)\right) \\
&= \lim_{x\to\infty} \arctan\left(\infty\left(\frac{1}{\infty} - 1\right)\right) \\
&= \lim_{x\to\infty} \arctan\big(\infty(-1)\big) && \left(\frac{1}{\infty} = 0\right) \\
&= \lim_{x\to\infty} \arctan(-\infty)
\end{aligned}
$$

If we recall our arctan function graph, we see that as $x \to -\infty$ gives us $-\frac{\pi}{2}$.

$\therefore \lim_{x\to\infty} \arctan(x^2 - x^4) = -\frac{\pi}{2}$

Strategy: Squeeze Theorem for Bounded functions

Squeeze Theorem:

Given:

1) **If $f(x)$ is stuck between two other functions $L(x) \le f(x) \le H(x)$**
2) $\lim_{x\to a} L(x) = \lim_{x\to a} H(x) = K$

Conclusion:

$\lim_{x\to a} f(x) = K$

How To Use it:

1) **Find an expression for that is higher than $f(x)$. Call this $H(x)$.**
2) **Find an expression that is lower than $f(x)$. Call this $L(x)$.**
3) **Show that the limits for $H(x)$ and $L(x)$ go to the same value K.**
4) **Conclude (by squeeze theorem) that $f(x)$ also goes to value k.**

When To Use it:

If the expression has terms that have a max and a min over the whole domain, (such as sin and cos), then we usually try using squeeze theorem.

Why this works?

If both the far left side and the far right side both go to K, then we are stuck with the inequality:

$$K \le \lim_{x\to a} f(x) \le \mathrm{K}$$

Which forces the limit to be K as that is the only number between K and K.

Examples: Squeeze Theorem

Example 12:

Find the value of the limit: $\lim_{x\to\infty} \frac{\sin(x)}{x}$

Solution:

Careful to not confuse this with $\lim_{x\to 0} \frac{\sin(x)}{x}$

We note that as $x \to \infty$, the sin function jumps back and forth between 1 and -1. Thus this function (on its own) does not exist. However, we do know that $-1 \leq \sin(x) \leq 1$, so we can use this to bound the function:

Since $L(x) = -\frac{1}{x} \leq \frac{\sin(x)}{x} \leq \frac{1}{x} = H(x)$

Then we get $\lim_{x\to\infty} -\frac{1}{x} = 0 = \lim_{x\to\infty} \frac{1}{x}$ we have $\lim_{x\to\infty} \frac{\sin(x)}{x} = 0$ by the squeeze theorem.

Examples: Squeeze Theorem

Example 13:

Find the value of the limit: $\lim\limits_{x\to\infty} \frac{3\sin(x)-2\cos(x)}{e^x}$

Solution:

We note that as $x \to \infty$, the sin and cos functions jump back and forth between 1 and -1. Thus these functions (on their own) do not exist. However, we do know that $-1 \le \sin(x) \le 1$ along with $-1 \le \cos(x) \le 1$, so we can use this to bound the function:

$-1 \le \sin(x) \le 1 \quad \to \quad -3 \le 3\sin(x) \le 3$

$-1 \le \cos(x) \le 1 \quad \to \quad 2 \ge -2\cos(x) \ge -2$ (multiplying by a negative)

$\to \quad -2 \le -2\cos(x) \le 2$ (rewriting the expression)

If we add them together we get $-5 \le 3\sin(x) - 2\cos(x) \le 5$ which gives

$$L(x) = -\frac{5}{e^x} \le \frac{3\sin(x)-2\cos(x)}{e^x} \le \frac{5}{e^x} = H(x)$$

Then we get $\lim\limits_{x\to\infty} -\frac{5}{e^x} = 0 = \lim\limits_{x\to\infty} \frac{5}{e^x}$ we have that our original limit $\lim\limits_{x\to 0} \frac{3\sin(x)-2\cos(x)}{e^x} = 0$ by the squeeze theorem.

Strategy: Using Substitution for limits going to $-\infty$

How To Use it:	When To Use it:	Why this works?
1) Let $y = -x$ then as $x \to -\infty$ we get $y \to \infty$. 2) Replace all $x = -y$ in the limit 3) Use another limit technique to solve the limit.	When the limit has $x \to -\infty$, we do a substitution first to make the limit look like our other limits.	Although you do not always have to do this, having a limit as $x \to \infty$ instead of $x \to -\infty$ avoids many subtle arithmetic errors.

Example 14:

Find the value of the limit: $\lim_{x\to-\infty} x + \sqrt{x^2 + x}$

Solution:

Let $y = -x$ and so as $x \to -\infty$ we have $y \to \infty$. Subbing in $x = -y$ we get:

$$\lim_{x\to-\infty} x + \sqrt{x^2 + x} \quad \to \lim_{y\to\infty} -y + \sqrt{(-y)^2 - y}$$

$$= \lim_{y\to\infty} \sqrt{y^2 - y} - y \quad (\infty - \infty \text{ with roots } \therefore \text{ conjugate})$$

$$= \lim_{y\to\infty} \left(\sqrt{y^2 - y} - y\right)\left(\frac{\sqrt{y^2-y}+y}{\sqrt{y^2-y}+y}\right)$$

$$= \lim_{y\to\infty} \frac{y^2-y-y^2}{\sqrt{y^2-y}+y}$$

$$= \lim_{y\to\infty} \frac{-y}{\sqrt{y^2-y}+y} \quad \left(\frac{\infty}{\infty} \text{ so use the high pow den: } y\right)$$

$$= \lim_{y\to\infty} \frac{-y}{\sqrt{y^2-y}+y} \div \frac{y}{y}$$

$$= \lim_{y\to\infty} \frac{-\frac{y}{y}}{\sqrt{\frac{y^2}{y^2}-\frac{y}{y^2}}+\frac{y}{y}} \qquad \left(\text{Inside a root, we note that } y = \sqrt{y^2}\right)$$

$$= \lim_{y\to\infty} \frac{-1}{\sqrt{1-\frac{1}{y}}+1}$$

$$= \frac{-1}{\sqrt{1-0}+1}$$

$$= -\frac{1}{2}$$